1. [**15 points**] For the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ find a matrix P such that PAP^{-1} is diagonal and give a formula for A^{30}.

Solution: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, then the characteristic polynomial of A is $P(A) = (x-2)^2 - 1 = x^2 - 4x - 3 = (x-3)(x-1)$, implies eigenvalues of A are 3 and 1.
Let (a, b) is an eigenvector corresponding to the eigenvalues 3. Then
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{matrix} 2a + b = 3a \\ a + 2b = 3b \end{matrix} \implies a = b.$
Thus $(1, 1)$ is an eigenvector corresponding to eigenvalue 3. Similarly if (c, d) is an eigenvector corresponding to the eigenvalue 1. Then
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 1 \begin{pmatrix} c \\ d \end{pmatrix} \implies \begin{matrix} 2c + d = c \\ c + 2d = d \end{matrix} \implies c = -d.$
Thus $(-1, 1)$ is an eigenvector corresponding to eigenvalue 1.
Now consider the matrix $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Also

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = P^{-1}DP \implies D = PAP^{-1},$$

where $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix.

Since $A = P^{-1}DP$, then $A^{30} = \underbrace{(P^{-1}DP) \cdots (P^{-1}DP)}_{30 \text{ times}} = P^{-1}D^{30}P$.

2. [**15 points**] Let F be a field and $n > 0$ an integer. Give an example of two $n \times n$ matrices A, B that are not similar, but whose characteristic polynomials are equal.

Solution: Assume that $F = \mathbb{C}$ and $n = 2$. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Then, A and B both have same characteristic polynomial x^2. But they are not similar, as $rank(A) = rank \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 \neq 0 = rank(B) = rank \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. [**15 points**] Let F be a field. Given two linearly independent collections of vectors $v_1, \cdots, v_r \in F^n$ and $w_1, \cdots, w_r \in F^n$, show that there exists an invertible $n \times n$ matrix A such that $Av_i = w_i$, for $i \leq r$.

Solution: Since $\{v_1, \cdots, v_r\} \subset F^n$ and $\{w_1, \cdots, w_r\} \subset F^n$ are linearly independent set of vectors, they can be extend to ordered bases for F^n.
Let $\mathcal{B} = \{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ and $\mathcal{B}' = \{w_1, \cdots, w_r, w_{r+1}, \cdots, w_n\}$ are bases for F^n. Consider a map $T : F^n \to F^n$ such that $T\Big(\sum_{i=1}^n a_i v_i\Big) = \sum_{i=1}^n a_i w_i$, for all scalar $\{a_1, \cdots, a_n\} \subset F$. Then T is linear as $T(\alpha v + \beta w) = T\Big(\alpha \sum_{i=1}^n a_i v_i + \beta \sum_{i=1}^n b_i v_i\Big) = T\Big(\sum_{i=1}^n (\alpha a_i + \beta b_i) v_i\Big) = \sum_{i=1}^n (\alpha a_i + \beta b_i) w_i = \alpha T(v) +$

$\beta T(w)$. Also T is bijective as $T(v) = T\Big(\sum_{i=1}^n a_i v_i\Big) = 0 \implies \sum_{i=1}^n a_i w_i = 0 \implies a_i = 0$ in F, $\forall\, i \in \{1, \cdots, n\} \implies v = 0 \implies nullity(T) = 0 \implies rank(T) = dim(F^n) - nullity(T) = n - 0 = n$.
Now let $A = [T]_{\mathcal{B}'}^{\mathcal{B}}$ be the matrix of T relative to the ordered bases $\mathcal{B}, \mathcal{B}'$. Then A is the $n \times n$ invertible matrix such that $Av_i = w_i$, for $i \le r$.

4. [20 points] Let A be an $m \times n$ matrix and let X be an invertible $m \times m$ matrix. Set $A' = XA$.
(i) Show that the columns $C_{i_1}, \cdots, C_{i_r}$ of A are linearly independent iff the corresponding columns $C'_{i_1}, \cdots, C'_{i_r}$ of A' are so. Likewise, show that $C_{i_1}, \cdots, C_{i_r}$ span the the column space of A iff the corresponding ones of A' have the same property.
(ii) Deduce that if the pivot entries in (reduced) row echelon form of A are in columns indexed by $j_1, \cdots, j_r$, then the corresponding columns $C_{j_1}, \cdots, C_{j_r}$ of A form a basis of the column space of A.

Solution: Since $A' = XA$, where X is $m \times m$ invertible matrix, then $C'_{i_j} = XC_{i_j}, \forall\, j \in \{1, \cdots, r\} \implies C_{i_j} = X^{-1}C'_{i_j}, \forall\, j \in \{1, \cdots, r\}$.

(i) Let $C_{j_1}, \cdots, C_{j_r}$ are linearly independent.
Thus, $0 = \sum_{j=1}^r a_{i_j} C'_{i_j} = \sum_{j=1}^r a_{i_j} XC_{i_j} = X\Big(\sum_{j=1}^r a_{i_j} C_{i_j}\Big) \implies \sum_{j=1}^r a_{i_j} C_{i_j} = X^{-1}0 = 0 \implies a_{i_j} = 0$, $\forall j \in \{1, \cdots, r\}$, as $C_{i_1}, \cdots, C_{i_r}$ are linearly independent, implies $C'_{i_1}, \cdots, C'_{i_r}$ are linearly independent.
Conversely let $C'_{i_1}, \cdots, C'_{i_r}$ are linearly independent.
Thus, $0 = \sum_{j=1}^r b_{i_j} C_{i_j} = \sum_{j=1}^r b_{i_j} X^{-1} C'_{i_j} = X^{-1}\Big(\sum_{j=1}^r b_{i_j} C'_{i_j}\Big) \implies \sum_{j=1}^r b_{i_j} C'_{i_j} = X0 = 0 \implies b_{i_j} = 0$, $\forall j \in \{1, \cdots, r\}$, as $C'_{i_1}, \cdots, C'_{i_r}$ are linearly independent, implies $C_{i_1}, \cdots, C_{i_r}$ are linearly independent.
Now let $C(A)$ and $C(A')$ be the corresponding column space of A and A'.
Let $C(A) = span\{C_{i_1}, \cdots, C_{i_r}\}$. Then, $v \in C(A') \implies X^{-1}v \in C(A) \implies X^{-1}v = \sum_{j=1}^r a_{i_j} C_{i_j} = \sum_{j=1}^r a_{i_j} X^{-1} C'_{i_j} = X^{-1}\Big(\sum_{j=1}^r a_{i_j} C'_{i_j}\Big) \implies v = \sum_{j=1}^r a_{i_j} C'_{i_j}$. Thus, $\{C'_{i_1}, \cdots, C'_{i_r}\}$ spans $C(A')$.
Conversely, let $C(A') = span\{C'_{i_1}, \cdots, C'_{i_r}\}$. Therefore, $v \in C(A') \implies Xv \in C(A) \implies Xv = \sum_{j=1}^r a_{i_j} C'_{i_j} = \sum_{j=1}^r a_{i_j} XC_{i_j} = X\Big(\sum_{j=1}^r a_{i_j} C_{i_j}\Big) \implies v = \sum_{j=1}^r a_{i_j} C_{i_j}$. Thus, $\{C_{i_1}, \cdots, C_{i_r}\}$ spans $C(A)$.

(ii) Let A' be the (reduced) row echelon form of A. Then $A' = (E_k \cdots E_1)A = PA$, where E_j, $j \in \{1, \cdots, k\}$ are $m \times m$ elementary matrices. Since E_j, $j \in \{1, \cdots, k\}$ are invertible, $P = E_k \cdots E_1$ is invertible. Thus if the pivot entries in A' are in columns indexed by $j_1, \cdots, j_r$, then $\{C'_{j_1}, \cdots, C'_{j_r}\}$ are linearly independent and spans $C(A')$. Therefore using previous result in (i) we get $\{C_{j_1}, \cdots, C_{j_r}\}$ form a basis of $C(A)$.

5. [15 points] Show that the equation $AB - BA = I$ has no solution in $n \times n$ matrices A, B over $\mathbb{R}$. Show that over the finite field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, there is a solution for A, B for $n = 2$.

Solution: Suppose there exist $n \times n$ matrices A, B over $\mathbb{R}$ such that $AB - BA = I$.
Then $trace(AB - BA) = trace(I) \implies trace(AB) - trace(BA) = trace(I) \implies 0 = n$ (as $trace(AB) = trace(BA)$), a contradiction. Hence, the equation $AB - BA = I$ has no solution in $n \times n$ matrices A, B over $\mathbb{R}$.
Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over the field $\mathbb{F}_2$. Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus

$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, as $-1 = 1$ in $\mathbb{F}_2$. Thus $AB - BA = I$ has a solution over the finite field $\mathbb{F}_2$ for $n = 2$.

6. [20 points] Let $v_1, \cdots, v_r$ be mutually orthogonal (nonzero) vectors in $\mathbb{R}^n$.
(i) If W is the subspace $\Big\{w \in \mathbb{R}^n | v_i \cdot w = 0 \ \forall i\Big\}$ and $V = span(v_1, \cdots, v_r)$, then show that dimension of W is $n - r$ and that $V \bigoplus W = \mathbb{R}^n$.
(ii) Show that any collection of mutually orthogonal vectors in $\mathbb{R}^n$ is a part of a basis consisting of mutually orthogonal vectors.

Solution: (i) Since $v_1, \cdots, v_r$ are mutually nonzero orthogonal vectors in $\mathbb{R}^n$, $\sum_{i=1}^r a_i v_i = 0 \implies \Big(\sum_{i=1}^r a_i v_i\Big) \cdot v_i = 0, \forall i \in \{1, \cdots, r\} \implies a_i v_i \cdot v_i = 0 \implies a_i = 0, \forall i \in \{1, \cdots, r\}$, as $v_i \cdot v_i \neq 0$. Thus, $v_1, \cdots, v_r$ are linearly independent. So they can be extends to a basis for $\mathbb{R}^n$, say $\mathcal{B} = \{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ is the basis for $\mathbb{R}^n$. Then $v_i \cdot v_i \neq 0, \forall i \in \{1, \cdots, n\}$. Now using Gram-Schmidt orthogonalization process (by taking $w_1 = v_1$ and $w_i = v_i - \sum_{k=1}^{i-1} \frac{v_k \cdot w_k}{w_k \cdot w_k} w_k, \forall i \in \{2, \cdots, n\}$) $\mathcal{B}$ can be transformed into a orthogonal basis $\mathcal{B}' = \{w_1, \cdots, w_n\}$ for $\mathbb{R}^n$. Then $w_j = v_j, \forall j \in \{1, \cdots, r\}$; $w_i \cdot w_j = 0, \forall i \neq j$ and $\mathbb{R}^n = V \bigoplus W'$, where W' is the subspace spanned by $\{w_{r+1}, \cdots, w_n\}$.
Now let $w = \sum_{i=1}^n a_i w_i \in W$ then for any $v \in V$ $v \cdot w = 0 \implies v_j . \sum_{i=1}^n a_i w_i = 0, \forall j \in \{1, \cdots, r\} \implies a_j = 0, \forall j \in \{1, \cdots, r\} \implies w = \sum_{i=r+1}^n a_i w_i \in W' \implies W \subset W'$. Also since for any $w' \in W' \implies v \cdot w' = 0 \implies W' \subset W$. Thus $W = W'$, implies $\mathbb{R}^n = V \bigoplus W$ where $dim(W) = dim(W') = dim(\mathbb{R}^n) - dim(V) = n - r$.

(ii) Let $v_1, \cdots, v_r$ are mutually nonzero orthogonal vectors in $\mathbb{R}^n$, then they are linearly independent, so $r \leq dim(\mathbb{R}^n) = n$. If $r = n$ then $\{v_1, \cdots, v_r\}$ is the orthogonal basis. If not then by using same procedure in (i) i.e, extending it to a basis and then using Gram-Schmidt orthogonalization procedure we can get a orthogonal basis $\{v_1, \cdots, v_r, w_{r+1}, \cdots, w_n\}$ for $\mathbb{R}^n$.